

Array Orthogonality in Higher Dimensions

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ABSTRACT. We generalize the array orthogonality property for perfect autocorrelation sequences to n -dimensional arrays. The generalized array orthogonality property is used to derive a number of n -dimensional perfect array constructions.

1 Introduction

Heimiller [Heimiller, 1961] and Frank [Frank, 1962] introduced a construction for perfect sequences of length n^2 over n roots of unity. Heimiller proved the construction produces perfect sequences of prime lengths by relating the autocorrelation of the sequence to the autocorrelation and cross-correlation of the columns of an array *associated* with the sequence. Similarly, sequences constructed by Milewski [Milewski, 1983] used the same method to prove they were perfect. Mow [Mow, 1993] introduced the *array orthogonality property*, which generalized the proofs of Heimiller and Milewski to an arbitrary perfect sequence which is constructed by enumerating row-by-row the array associated with the sequence. Recently, the author [Blake, 2014] gave a sequence construction which possess the *array orthogonality property*.

2 Preliminaries

The *periodic cross-correlation* of the sequences, $\mathbf{a} = [a_0, a_1, \dots, a_{n-1}]$ and $\mathbf{b} = [b_0, b_1, \dots, b_{n-1}]$ for shift τ is defined as

$$\theta_{\mathbf{a}, \mathbf{b}}(\tau) = \sum_{i=0}^{n-1} a_i b_{i+\tau}^*,$$

where $i + \tau$ is computed modulo n . Two sequences are *orthogonal* if $\theta_{\mathbf{a}, \mathbf{b}}(\tau) = 0$ for all τ .

The *periodic autocorrelation* of a sequence, \mathbf{s} for shift τ is given by $\theta_{\mathbf{s}}(\tau) = \theta_{\mathbf{s}, \mathbf{s}}(\tau)$. For $\tau \neq 0 \bmod n$, $\theta_{\mathbf{s}}(\tau)$ is called an *off-peak* autocorrelation. A sequence is *perfect* if all off-peak periodic autocorrelation values are zero.

For applications, long perfect binary sequences are desired. However, the longest known perfect binary sequence is the length 4 sequence $[1, 1, 1, -1]$. It is conjectured that longer perfect binary sequences do not exist [Mow, 1993, conj. 3.9, pp. 49]. Consequently, sequences over roots of unity have been investigated for the last 60 years.

An N -dimensional array, \mathbf{S} , over n roots of unity is defined as

$$\mathbf{S} = [S_{i_0, i_1, \dots, i_{N-1}}] = \omega^{f(i_0, i_1, \dots, i_{N-1})},$$

where $f(i_0, i_1, \dots, i_{N-1})$ is an integer function and ω is a primitive n^{th} root of unity, that is $\omega = e^{2\pi\sqrt{-1}/n}$.

A sequence is simply a one-dimensional array. The periodic cross-correlation of two N -dimensional arrays, \mathbf{A} and \mathbf{B} , both of size $l_0 \times l_1 \times \dots \times l_{N-1}$, for shift s_0, s_1, \dots, s_{N-1} is defined as

$$\theta_{\mathbf{A}, \mathbf{B}}(s_0, s_1, \dots, s_{N-1}) = \sum_{i_0=0}^{l_0-1} \sum_{i_1=0}^{l_1-1} \dots \sum_{i_{N-1}=0}^{l_{N-1}-1} A_{i_0, i_1, \dots, i_{N-1}} B_{i_0+s_0, i_1+s_1, \dots, i_{N-1}+s_{N-1}}^*.$$

Similarly, the periodic autocorrelation of a N -dimensional array for shift s_0, s_1, \dots, s_{N-1} is given by $\theta_{\mathbf{A}}(s_0, s_1, \dots, s_{N-1}) = \theta_{\mathbf{A}, \mathbf{A}}(s_0, s_1, \dots, s_{N-1})$. $\theta_{\mathbf{A}}(s_0, s_1, \dots, s_{N-1})$ is called an *off-peak* autocorrelation if not all $s_i = 0 \bmod l_i$. An array is *perfect* if all off-peak autocorrelations are zero.

3 The Array Orthogonality Property

We begin with the array orthogonality property (AOP). Consider a sequence $\mathbf{s} = [s_0, s_1, \dots, s_{ld^2-1}]$, then we call

$$\mathbf{S} = [S_{i,j}] = \begin{bmatrix} s_0 & s_1 & s_2 & \cdots & \cdots & s_{d-1} \\ s_d & s_{d+1} & s_{d+2} & \cdots & \cdots & s_{2d-1} \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ s_{(l-1)d} & s_{(l-1)d+1} & s_{(l-1)d+2} & \cdots & \cdots & s_{dl-1} \end{bmatrix}$$

the array *associated* with \mathbf{s} for the *divisor* d . We use the notation $\mathbf{S}[n]$ to denote the n -th column of \mathbf{S} .

Definition 3.1 (AOP). [Mow, 1993] A sequence $\mathbf{s} = [s_0, s_1, \dots, s_{ld^2-1}]$ has the AOP for the divisor d if the array \mathbf{S} associated with \mathbf{s} has the following two properties:

1. For all τ and $j_0 \neq j_1 \bmod d$: $\theta_{\mathbf{S}[j_0], \mathbf{S}[j_1]}(\tau) = 0$. (That is, any two distinct columns of \mathbf{S} are orthogonal.)
2. For all $\tau \neq 0 \bmod ld$: $\sum_{j=0}^{d-1} \theta_{\mathbf{S}[j]}(\tau) = 0$. (That is, the columns of \mathbf{S} form a set of periodic complementary sequences.)

Example 3.2. We show that the Frank-Heimiller sequence of length 16 over 4 roots of unity has the AOP for the divisor $d = 4$. The sequence, in index notation (that is, the mapping: $2\pi\sqrt{-1}s_n/4 \rightarrow s_n$), is given by

$$\mathbf{s} = [0, 0, 0, 0, 0, 1, 2, 3, 0, 2, 0, 2, 0, 3, 2, 1],$$

and the array, \mathbf{S} , associated with \mathbf{s} for the divisor 4 is given by

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix}.$$

The cross-correlation of all 6 distinct pairs of columns is given by

$$\theta_{[0,0,0,0],[0,1,2,3]} = \theta_{[0,0,0,0],[0,2,0,2]} = \theta_{[0,0,0,0],[0,3,2,1]} = \theta_{[0,1,2,3],[0,2,0,2]} = \theta_{[0,1,2,3],[0,3,2,1]} = \theta_{[0,2,0,2],[0,3,2,1]} = [0, 0, 0, 0].$$

So all distinct pairs of columns of \mathbf{S} are orthogonal. Thus, \mathbf{s} satisfies the first condition of the AOP. We now compute the autocorrelation of all the columns of \mathbf{S} :

$$\begin{aligned} \theta_{[0,0,0,0]} &= [4, 4, 4, 4] \\ \theta_{[0,1,2,3]} &= [4, 4\sqrt{-1}, -4, -4\sqrt{-1}] \\ \theta_{[0,2,0,2]} &= [4, -4, 4, -4] \\ \theta_{[0,3,2,1]} &= [4, -4\sqrt{-1}, -4, 4\sqrt{-1}] \end{aligned}$$

For each off-peak shift, the sum of the autocorrelations of all the columns of \mathbf{S} is zero. Thus, \mathbf{s} satisfies the second condition of the AOP.

Theorem 3.3. [Mow, 1993] Any sequence with the AOP is perfect.

Proof. The periodic autocorrelation of a sequence \mathbf{s} , of length ld^2 for shift τ is given by

$$\theta_{\mathbf{s}}(\tau) = \sum_{i=0}^{ld^2-1} s_i s_{i+\tau}^*.$$

Change coordinates, let $i = qd + r$, ($r < d$), and $\tau = q'd + r'$, ($r' < d$). Then we have

$$\begin{aligned} \theta_{\mathbf{s}}(q'd + r') &= \sum_{r=0}^{d-1} \sum_{q=0}^{ld-1} s_{qd+r} s_{(q+q')d+r+r'}^* \\ &= \sum_{r=0}^{d-1} \sum_{q=0}^{ld-1} s_{qd+r} s_{(q+q'+\lfloor \frac{r+r'}{d} \rfloor)d+(r+r' \bmod d)}^* \\ &= \sum_{r=0}^{d-1} \sum_{q=0}^{ld-1} s_{q,r} s_{q+q'+\lfloor \frac{r+r'}{d} \rfloor, (r+r' \bmod d)}^* \\ &= \sum_{r=0}^{d-1} \theta_{\mathbf{S}[r], \mathbf{S}[r+r' \bmod d]} \left(q' + \left\lfloor \frac{r+r'}{d} \right\rfloor \right). \end{aligned}$$

For $r' \neq 0$, condition 1 of the AOP implies $\theta_{\mathbf{s}}(\tau) = 0$. Otherwise, for $r' = 0$, condition 2 of the AOP implies $\theta_{\mathbf{s}}(\tau) = 0$. ■

The Frank and Heimpler sequences were the first sequences constructed which possessed the AOP.

CONSTRUCTION I [Heimpler, 1961][Frank, 1962] We construct a sequence \mathbf{s} of length n^2 over n roots of unity. Let $\mathbf{S}' = [S'_{i,j}] = \omega^{ij}$ be an $n \times n$ array where $\omega = e^{2\pi\sqrt{-1}/n}$. The sequence \mathbf{s} is constructed by enumerating row-by-row the array \mathbf{S}' .

Heimpler showed \mathbf{s} is perfect by showing \mathbf{S}' had the AOP. Heimpler's construction had the restriction that n be a prime number. Frank generalized the Heimpler construction by removing this restriction. Other sequence constructions with the AOP include Milewski sequences [Milewski, 1983] and constructions by the author [Blake, 2014].

CONSTRUCTION II [Milewski, 1983] We construct a perfect sequence, \mathbf{s} , of length m^{2k+1} over m^{k+1} roots of unity, where $k \geq 1$. Let $\mathbf{u} = [u_i]$ be a Chu sequence [Chu, 1972] of length m . Let $\mathbf{S}' = [S'_{i,j}] = u_{i \bmod m} \omega^{ij}$ be a $m^{k+1} \times m^k$ array where $\omega = e^{2\pi\sqrt{-1}/m^{k+1}}$. The sequence \mathbf{s} is constructed by enumerating row-by-row the array \mathbf{S}' .

The following construction borrows elements of the constructions of Frank and Milewski. The idea of using a piecewise function within a perfect sequence construction was introduced by Liu and Fan [Liu, 2004].

CONSTRUCTION III [Blake, 2014] We construct a perfect sequence of length $4mn^{k+1}$ over $2mn^k$ roots of unity. Let $\mathbf{S}' = [S'_{i,j}] = \omega^{\lfloor i(i+j)/n \rfloor}$ be a $2mn^{k+1} \times 2$ array over $2mn^k$ roots of unity, where $\omega = e^{2\pi\sqrt{-1}/(2mn^k)}$. The sequence \mathbf{s} is constructed by enumerating row-by-row the array \mathbf{S}' .

Sequence constructions which do not have the AOP include Chu sequences [Chu, 1972] and Liu-Fan sequences [Liu, 2004]. We use these constructions within the higher dimensional constructions. Sequences with the AOP are yet to be used within perfect sequence constructions.

4 The Generalized Array Orthogonality Property

The idea that the AOP may be used to construct arrays in higher dimensions was used by Blake et al [Blake, 2013]. We now turn our attention to AOP in higher dimensions. Consider two dimensions, let $\mathbf{A} = [A_{i,j}]$ be an $n \times m$ array. Then the array, \mathbf{A}' is given by

$$\left[\begin{array}{c} \begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,d-1} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{d-1,0} & A_{d-1,1} & \cdots & A_{d-1,d-1} \end{bmatrix} \\ \begin{bmatrix} A_{d,0} & A_{d,1} & \cdots & A_{d,d-1} \\ \vdots & \vdots & & \vdots \\ & & \ddots & \\ A_{2d-1,0} & A_{2d-1,1} & \cdots & A_{2d-1,d-1} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} A_{n-d-1,0} & A_{n-d-1,1} & \cdots & A_{n-d-1,d-1} \\ A_{n-d,0} & \vdots & & \vdots \\ & & \ddots & \\ A_{n-1,0} & A_{n-1,1} & \cdots & A_{n-1,d-1} \end{bmatrix} \end{array} \right] \cdots \left[\begin{array}{c} \begin{bmatrix} A_{0,m-d-1} & A_{0,m-d} & \cdots & A_{0,m-1} \\ A_{1,m-d-1} & A_{1,m-d} & \cdots & A_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{d-1,m-d-1} & A_{d-1,m-d} & \cdots & A_{d-1,m-1} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} A_{n-d-1,m-d-1} & A_{n-d-1,m-d} & \cdots & A_{n-d-1,m-1} \\ A_{n-d,m-d-1} & A_{n-d,m-d} & \cdots & A_{n-d,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1,m-d-1} & A_{n-1,m-d} & \cdots & A_{n-1,m-1} \end{bmatrix} \end{array} \right].$$

We call this array the array *associated* with \mathbf{A} for the *divisor* d . We use the notation $\mathbf{A}'[k,l]$ to index $A_{i,j,k,l}$ for all i,j . We now state the n -dimensional generalization of array association.

Definition 4.1 (Array association). Let \mathbf{A}' the $2n$ -dimensional array $\mathbf{A}' = [A'_{i_0,i_1,\dots,i_{2n-1}}]$, then the n -dimensional array \mathbf{A} associated with \mathbf{A}' for the divisor d is given by

$$\mathbf{A} = [A'_{di_0+i_n, di_1+i_{n+1}, \dots, di_{n-1}+i_{2n-1}}].$$

We can now state the *generalized array orthogonality property* (GAOP).

Definition 4.2 (GAOP). An n -dimensional array, \mathbf{A} , has the GAOP for the divisor d if the $2n$ -dimensional array \mathbf{A}' associated with \mathbf{A} has the following properties:

1. For all s_0, s_1, \dots, s_{n-1} and for all $i_n, i_{n+1}, \dots, i_{2n-1}, j_n, j_{n+1}, \dots, j_{2n-1} \bmod d$ such that $(i_n, i_{n+1}, \dots, i_{2n-1}) \neq (j_n, j_{n+1}, \dots, j_{2n-1})$:

$$\theta_{\mathbf{A}'[i_n, i_{n+1}, \dots, i_{2n-1}], \mathbf{A}'[j_n, j_{n+1}, \dots, j_{2n-1}]}(s_0, s_1, \dots, s_{n-1}) = 0.$$

(That is, all distinct n -dimensional arrays of \mathbf{A}' are orthogonal.)

2. For $s_0, s_1, \dots, s_{n-1} \bmod d$ such that not all $s_i = 0 \bmod d$ (off-peak autocorrelation):

$$\sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \cdots \sum_{i_{n-1}=0}^{d-1} \theta_{\mathbf{A}'[i_n, i_{n+1}, \dots, i_{2n-1}]}(s_0, s_1, \dots, s_{n-1}) = 0.$$

(That is, all the arrays $\mathbf{A}'[i_n, i_{n+1}, \dots, i_{2n-1}]$ form a set of periodic complementary arrays.)

We now state and prove our main theorem.

Theorem 4.3. *Any n -dimensional array with the GAOP is perfect.*

Proof. Consider the autocorrelation of the array $\mathbf{A} = [A_{i_0, i_1, \dots, i_{n-1}}]$, with size $m_0 \times m_1 \times \dots \times m_{n-1}$,

$$\theta_{\mathbf{A}}(s_0, s_1, \dots, s_{n-1}) = \sum_{i_0=0}^{m_0-1} \sum_{i_1=0}^{m_1-1} \dots \sum_{i_{n-1}=0}^{m_{n-1}-1} A_{q_0 d + r_0, \dots, q_{n-1} d + r_{n-1}} A_{i_0 + s_0, i_1 + s_1, \dots, i_{n-1} + s_{n-1}}^*$$

Introduce the change of variables $i_k = q_k d + r_k$, ($r_k < d$), then we have

$$\theta_{\mathbf{A}}(s_0, s_1, \dots, s_{n-1}) = \sum_{r_0=0}^{d-1} \dots \sum_{r_{n-1}=0}^{d-1} \sum_{q_0=0}^{m_0/d-1} \dots \sum_{q_{n-1}=0}^{m_{n-1}/d-1} A_{q_0 d + r_0, \dots, q_{n-1} d + r_{n-1}} A_{q_0 d + r_0 + s_0, \dots, q_{n-1} d + r_{n-1} + s_{n-1}}^*.$$

As before, introduce the change of variables $s_k = q'_k d + r'_k$, ($r'_k < d$), then we have

$$\begin{aligned} \theta_{\mathbf{A}}(s_0, s_1, \dots, s_{n-1}) &= \sum_{r_0=0}^{d-1} \dots \sum_{r_{n-1}=0}^{d-1} \sum_{q_0=0}^{m_0/d-1} \dots \sum_{q_{n-1}=0}^{m_{n-1}/d-1} A_{q_0 d + r_0, \dots, q_{n-1} d + r_{n-1}} A_{(q_0 + q'_0) d + r_0 + r'_0, \dots, (q_{n-1} + q'_{n-1}) d + r_{n-1} + r'_{n-1}}^* \\ &= \sum_{r_0=0}^{d-1} \dots \sum_{r_{n-1}=0}^{d-1} \sum_{q_0=0}^{m_0/d-1} \dots \sum_{q_{n-1}=0}^{m_{n-1}/d-1} A_{q_0 d + r_0, \dots, q_{n-1} d + r_{n-1}} \times \\ &\quad A_{\left(q_0 + q'_0 \left\lfloor \frac{r_0 + r'_0}{d} \right\rfloor\right) d + (r_0 + r'_0 \bmod d), \dots, \left(q_{n-1} + q'_{n-1} \left\lfloor \frac{r_{n-1} + r'_{n-1}}{d} \right\rfloor\right) d + (r_{n-1} + r'_{n-1} \bmod d)}^*. \end{aligned}$$

Let \mathbf{A}' be a $2n$ -dimensional array with size $m_0/d \times m_1/d \times \dots \times m_{n-1}/d \times d \times d \times \dots \times d$. (\mathbf{A}' the array associated with \mathbf{A} .) Then we have

$$\begin{aligned} \theta_{\mathbf{A}}(s_0, s_1, \dots, s_{n-1}) &= \sum_{r_0=0}^{d-1} \dots \sum_{r_{n-1}=0}^{d-1} \sum_{q_0=0}^{m_0/d-1} \dots \sum_{q_{n-1}=0}^{m_{n-1}/d-1} A'_{q_0, \dots, q_{n-1}, r_0, \dots, r_{n-1}} \times \\ &\quad A'^*_{q_0 + q'_0 \left\lfloor \frac{r_0 + r'_0}{d} \right\rfloor, \dots, q_{n-1} + q'_{n-1} \left\lfloor \frac{r_{n-1} + r'_{n-1}}{d} \right\rfloor, r_0 + r'_0 \bmod d, \dots, r_{n-1} + r'_{n-1} \bmod d} \\ &= \sum_{r_0=0}^{d-1} \dots \sum_{r_{n-1}=0}^{d-1} \theta_{\mathbf{A}'[r_0, \dots, r_{n-1}], \mathbf{A}'[r_0 + r'_0 \bmod d, \dots, r_{n-1} + r'_{n-1} \bmod d]}(Q_0, Q_1, \dots, Q_{n-1}), \end{aligned}$$

where $Q_k = q_k + q'_k \left\lfloor \frac{r_k + r'_k}{d} \right\rfloor$, and $\mathbf{A}'[i_0, i_1, \dots, i_{n-1}]$ is the n -dimensional array $[A'_{i_0, i_1, \dots, i_{2n-1}}]$ where $i_n, i_{n+1}, \dots, i_{2n-1}$ are fixed for each array. When $r_0 + r'_0 = r_1 + r'_1 = \dots = r_{n-1} + r'_{n-1} = 0 \bmod d$ condition 1 of the GAOP implies $\theta_{\mathbf{A}}(s_0, s_1, \dots, s_{n-1}) = 0$. Otherwise, condition 2 of the GAOP implies $\theta_{\mathbf{A}}(s_0, s_1, \dots, s_{n-1}) = 0$. \blacksquare

Note that the divisor, d , does not have to be the same in each dimension. Furthermore, as is the case in one-dimension, the array \mathbf{A}' is perfect.

Corollary 4.4. *The array \mathbf{A}' is perfect.*

Proof. The proof follows from the fact that \mathbf{A} has the GAOP. \blacksquare

We now show the value of the GAOP by stating a construction for perfect m -dimensional arrays which are constructed by concatenating (perfect) $2m$ -dimensional arrays.

CONSTRUCTION IV Let

$$\mathbf{S}' = [S'_{i_0, i_1, \dots, i_{2m-1}}] = \omega \left(\prod_{n=m}^{2m-1} i_n + \sum_{n=0}^{m-1} i_n i_{n+m} \right)$$

be a $2m$ -dimensional array of size $d \times d \times \dots \times d$, where $\omega = e^{2\pi \sqrt{-1}/d}$. Let \mathbf{S} be the m -dimensional array of size $d^2 \times d^2 \times \dots \times d^2$, formed by concatenating the array \mathbf{S}' .

Construction IV can be thought of a multi-dimensional generalization of Heimpler–Frank sequences. For $m = 1$, Construction IV produces Heimpler–Frank sequences.

Example 4.5. We show a 9×9 array, \mathbf{S} , from Construction IV has the GAOP for the divisor $d = 3$. The array \mathbf{S} , (in index notation), is given by

$$\mathbf{S} = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 2 & 0 & 1 & 2 & 1 & 0 & 2 \\ 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 2 & 2 & 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 & 0 & 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the array \mathbf{S}' associated with \mathbf{S} for the divisor 3 is given by

$$\mathbf{S}' = \left[\begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \\ 2 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right].$$

We show the arrays $\mathbf{S}'[1, 1]$, and $\mathbf{S}'[0, 2]$ are orthogonal. The arrays are given by

$$\mathbf{S}'[1, 1] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{S}'[0, 2] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

and their cross-correlation, for all shifts, is given by

$$\theta_{\mathbf{S}'[1,1], \mathbf{S}'[0,2]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The full calculation of the cross-correlation of all 36 distinct pairs of arrays is given in Appendix I. The sum of the correlations of all arrays $\mathbf{S}'[i, j]$, for $0 \leq i < 3$ and $0 \leq j < 3$ is given by

$$\sum_{i,j} \theta_{\mathbf{S}'[i,j]} = \begin{bmatrix} 81 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So \mathbf{S} satisfies the second condition of the GAOP.

Theorem 4.6. *The array \mathbf{S} from Construction IV is perfect.*

Proof. We show the array, $\mathbf{S} = [S_{i_0, i_1, \dots, i_{2m-1}}]$, is perfect by showing \mathbf{S} has the GAOP. Firstly, we show that all distinct m -dimensional arrays of \mathbf{S}' are orthogonal.

$$\begin{aligned}
& \theta_{\mathbf{S}'[i_m, i_{m+1}, \dots, i_{2m-1}], \mathbf{S}'[i'_m, i'_{m+1}, \dots, i'_{2m-1}]}(s_0, s_1, \dots, s_{m-1}) = \\
& \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{m-1}=0}^{d-1} S'_{i_0, i_1, \dots, i_{m-1}, i_m, i_{m+1}, \dots, i_{2m-1}} S'^*_{i_0+s_0, i_1+s_1, \dots, i_{m-1}+s_{m-1}, i'_m, i'_{m+1}, \dots, i'_{2m-1}} \\
& = \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{m-1}=0}^{d-1} \omega^{(\prod_{n=m}^{2m-1} i_n + \sum_{n=0}^{m-1} i_n i_{n+m})} \omega^{-(\prod_{n=m}^{2m-1} i'_n + \sum_{n=0}^{m-1} (i_n + s_n) i'_{n+m})} \\
& = \left(\omega^{\prod_{n=m}^{2m-1} i_n - \prod_{n=m}^{2m-1} i'_n - \sum_{n=0}^{m-1} s_n i'_{n+m}} \right) \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{m-1}=0}^{d-1} \omega^{\sum_{n=0}^{m-1} (i_{n+m} - i'_{n+m}) i_n} \\
& = \left(\omega^{\prod_{n=m}^{2m-1} i_n - \prod_{n=m}^{2m-1} i'_n - \sum_{n=0}^{m-1} s_n i'_{n+m}} \right) \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{m-1}=0}^{d-1} \left(\prod_{n=0}^{m-1} \omega^{(i_{n+m} - i'_{n+m}) i_n} \right) \\
& = \left(\omega^{\prod_{n=m}^{2m-1} i_n - \prod_{n=m}^{2m-1} i'_n - \sum_{n=0}^{m-1} s_n i'_{n+m}} \right) \left(\sum_{i_0=0}^{d-1} \omega^{(i_m - i'_m) i_0} \right) \left(\sum_{i_1=0}^{d-1} \omega^{(i_{m+1} - i'_{m+1}) i_1} \right) \times \dots \\
& \quad \times \left(\sum_{i_{m-1}=0}^{d-1} \omega^{(i_{2m-1} - i'_{2m-1}) i_{m-1}} \right)
\end{aligned}$$

The sums above are Gaussian sums, which are zero as $i_n \neq i'_n$ for all $m \leq n < 2m$. So \mathbf{S} satisfies the first condition of the GAOP. We now show \mathbf{S} satisfies the second condition of the GAOP.

$$\begin{aligned}
& \sum_{i=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{m-1}=0}^{d-1} \theta_{\mathbf{S}'[i_m, i_{m+1}, \dots, i_{2m-1}]}(s_0, s_1, \dots, s_{m-1}) = \\
& \sum_{i=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{2m-1}=0}^{d-1} S'_{i, i_1, \dots, i_{2m-1}} S'^*_{i_0+s_0, i_1+s_1, \dots, i_{m-1}+s_{m-1}, i_m, i_{m+1}, \dots, i_{2m-1}} \\
& = \sum_{i=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{2m-1}=0}^{d-1} \omega^{(\prod_{n=m}^{2m-1} i_n + \sum_{n=0}^{m-1} i_n i_{n+m})} \omega^{-(\prod_{n=m}^{2m-1} i_n + \sum_{n=0}^{m-1} (i_n + s_n) i_{n+m})} \\
& = \sum_{i=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{2m-1}=0}^{d-1} \omega^{-\sum_{n=0}^{m-1} s_n i_{n+m}} \\
& = \sum_{i=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{2m-1}=0}^{d-1} \left(\prod_{n=0}^{m-1} \omega^{-s_n i_{n+m}} \right) \\
& = \left(\sum_{i_m=0}^{d-1} \omega^{-s_0 i_m} \right) \left(\sum_{i_{m+1}=0}^{d-1} \omega^{-s_1 i_{m+1}} \right) \times \dots \times \left(\sum_{i_{2m-1}=0}^{d-1} \omega^{-s_{m-1} i_{2m-1}} \right)
\end{aligned}$$

Each of the above terms are Gaussian sums. At least one of the sums is zero, as we are computing off-peak autocorrelations. So \mathbf{S} satisfies the second condition of the GAOP. Thus, \mathbf{S} is a perfect array. \blacksquare

In one dimension, the largest known sequence with the AOP is a Frank–Heimiller sequence. The obvious question remains: are there GAOP-type constructions which build larger arrays than Construction IV? The

following construction produces arrays which are 4 times the size of Construction IV, but is restricted to two dimensions.

CONSTRUCTION V	Let
$\mathbf{S}' = [S'_{i_0, i_1, i_2, i_3}] = \omega \left\lfloor \frac{(d i_0 + i_2)(d i_1 + i_3)}{2d} \right\rfloor$	
<p>be a 4-dimensional array of size $2d \times 2d \times d \times d$, where $\omega = e^{2\pi\sqrt{-1}/d}$ and d is even. Let</p>	
$\mathbf{S} = [S_{i,j}] = \omega \left\lfloor \frac{i j}{2d} \right\rfloor$	
<p>be an array of size $2d^2 \times 2d^2$ formed by concatenating the array \mathbf{S}'.</p>	

Theorem 4.7. *The array \mathbf{S} from Construction V is perfect.*

Proof. We show the array \mathbf{S} is perfect by showing it has the GAOP for the divisor d . Firstly, we show that all distinct 2-dimensional arrays of \mathbf{S}' are orthogonal.

$$\begin{aligned}
\theta_{\mathbf{S}'[i_2, i_3], \mathbf{S}'[i'_2, i'_3]}(h, v) &= \sum_{i_0=0}^{2d-1} \sum_{i_1=0}^{2d-1} S'_{i_0, i_1, i_2, i_3} S'^*_{i_0+v, i_1+h, i_2, i_3} \\
&= \omega^{-dhv/2} \sum_{i_0=0}^{2d-1} \sum_{i_1=0}^{2d-1} \omega^{-\frac{d}{2}hi_0 - \frac{d}{2}vi_1 + \lfloor \frac{1}{2}(i_0i_3 + i_1i_2) + \frac{1}{2d}(i_2i_3) \rfloor - \lfloor \frac{1}{2}(i'_3i_0 + i'_2i_1 + i'_3v + i'_2h) + \frac{1}{2d}(i'_2i'_3) \rfloor}
\end{aligned}$$

We split the summation into i_0, i_1 even and odd. Consider the case when i_0, i_1 are even, then we have

$$\begin{aligned}
&\omega^{-dhv/2} \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \omega^{-\frac{d}{2}h(2i_0) - \frac{d}{2}v(2i_1) + \lfloor \mathcal{A} \rfloor - \lfloor \mathcal{B} \rfloor} \\
&= \omega^{-dhv/2 + \lfloor \mathcal{C} \rfloor - \lfloor \mathcal{D} \rfloor} \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \omega^{(i_3 - i'_3 - dh)i_0 + (i_2 - i'_2 - dv)i_1} \\
&= \omega^{-dhv/2 + \lfloor \mathcal{C} \rfloor - \lfloor \mathcal{D} \rfloor} \left(\sum_{i_0=0}^{d-1} \omega^{(i_3 - i'_3 - dh)i_0} \right) \left(\sum_{i_1=0}^{d-1} \omega^{(i_2 - i'_2 - dv)i_1} \right),
\end{aligned}$$

where $\mathcal{A} = \frac{1}{2}(2i_0i_3 + 2i_1i_2) + \frac{1}{2d}(i_2i_3)$, $\mathcal{B} = \frac{1}{2}(2i_0i'_3 + vi'_3 + 2i_1i'_2 + hi'_2) + \frac{1}{2d}(i'_2i'_3)$, $\mathcal{C} = \frac{1}{2d}(i_2i_3)$, and $\mathcal{D} = \frac{1}{2}(vi'_3 + hi'_2) + \frac{1}{2d}(i'_2i'_3)$. Both the Gaussian sums above are zero as $i_3 \neq i'_3$ and $i_2 \neq i'_2$. Now consider the case when i_0, i_1 are odd, then we have

$$\begin{aligned}
&\omega^{-dhv/2} \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \omega^{-\frac{d}{2}h(2i_0+1) - \frac{d}{2}v(2i_1+1) + \lfloor \mathcal{A} \rfloor - \lfloor \mathcal{B} \rfloor} \\
&= \omega^{-\frac{d}{2}(vh+h+v) + \lfloor \mathcal{C} \rfloor - \lfloor \mathcal{D} \rfloor} \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \omega^{(i_3 - i'_3 - dh)i_0 + (i_2 - i'_2 - dv)i_1} \\
&= \omega^{-\frac{d}{2}(vh+h+v) + \lfloor \mathcal{C} \rfloor - \lfloor \mathcal{D} \rfloor} \left(\sum_{i_0=0}^{d-1} \omega^{(i_3 - i'_3 - dh)i_0} \right) \left(\sum_{i_1=0}^{d-1} \omega^{(i_2 - i'_2 - dv)i_1} \right),
\end{aligned}$$

where $\mathcal{A} = \frac{1}{2}((2i_0+1)i_3 + (2i_1+1)i_2) + \frac{1}{2d}(i_2i_3)$, $\mathcal{B} = \frac{1}{2}((2i_0+1)i'_3 + vi'_3 + (2i_1+1)i'_2 + hi'_2) + \frac{1}{2d}(i'_2i'_3)$, $\mathcal{C} = \frac{1}{2}(i_3 + i_2) + \frac{1}{2d}(i_2i_3)$, and $\mathcal{D} = \frac{1}{2}(i'_3 + vi'_3 + i'_2 + hi'_2) + \frac{1}{2d}(i'_2i'_3)$. Both the Gaussian sums above are zero

as $i_3 \neq i'_3$ and $i_2 \neq i'_2$. Thus, \mathbf{S} satisfies the first condition of the GAOP.

We now show \mathbf{S} satisfies the second condition of the GAOP.

$$\begin{aligned}
\sum_{i_0=0}^{2d-1} \sum_{i_1=0}^{2d-1} \theta_{\mathbf{S}'[i_2, i_3]}(h, v) &= \sum_{i_0=0}^{2d-1} \sum_{i_1=0}^{2d-1} \sum_{i_2=0}^{d-1} \sum_{i_3=0}^{d-1} S'_{i_0, i_1, i_2, i_3} S'^*_{i_0+v, i_1+h, i_2, i_3} \\
&= \omega^{-\frac{d}{2}vh} \sum_{i_0=0}^{2d-1} \sum_{i_1=0}^{2d-1} \sum_{i_2=0}^{d-1} \sum_{i_3=0}^{d-1} \omega^{-\frac{d}{2}hi_0 - \frac{d}{2}vi_1 + \lfloor \mathcal{A} \rfloor - \lfloor \mathcal{B} \rfloor} \\
&= \omega^{-\frac{d}{2}vh} \sum_{i_0=0}^{2d-1} \sum_{i_1=0}^{2d-1} \sum_{i_2=0}^{d-1} \sum_{i_3=0}^{d-1} \omega^{-\frac{d}{2}hi_0 - \frac{d}{2}vi_1 + \lfloor \mathcal{A} \rfloor - \lfloor \mathcal{A} + \frac{1}{2}(hi_2 + vi_3) \rfloor},
\end{aligned}$$

where $\mathcal{A} = \frac{1}{2}(i_0i_3 + i_1i_2) + \frac{1}{2d}(i_2i_3)$ and $\mathcal{B} = \frac{1}{2}(i_0i_3 + i_1i_2 + hi_2 + vi_3) + \frac{1}{2d}(i_2i_3)$. We split the summation into i_2, i_3 even and odd. Consider the case when i_2, i_3 are even. Then we have

$$\begin{aligned}
&\omega^{-\frac{d}{2}vh} \sum_{i_0=0}^{2d-1} \sum_{i_1=0}^{2d-1} \sum_{i_2=0}^{d/2-1} \sum_{i_3=0}^{d/2-1} \omega^{-\frac{d}{2}hi_0 - \frac{d}{2}vi_1 + \lfloor \mathcal{C} \rfloor - \lfloor \mathcal{C} + \frac{1}{2}(2hi_2 + 2vi_3) \rfloor} \\
&= \omega^{-\frac{d}{2}vh} \sum_{i_0=0}^{2d-1} \sum_{i_1=0}^{2d-1} \sum_{i_2=0}^{d/2-1} \sum_{i_3=0}^{d/2-1} \omega^{-\frac{d}{2}hi_0 - \frac{d}{2}vi_1 - hi_2 - vi_3} \\
&= \omega^{-\frac{d}{2}vh} \left(\sum_{i_0=0}^{2d-1} \omega^{-\frac{d}{2}hi_0} \right) \left(\sum_{i_1=0}^{2d-1} \omega^{-\frac{d}{2}vi_1} \right) \left(\sum_{i_2=0}^{d/2-1} \omega^{-hi_2} \right) \left(\sum_{i_3=0}^{d/2-1} \omega^{-vi_3} \right),
\end{aligned}$$

where $\mathcal{C} = i_0i_3 + i_1i_2 + \frac{2}{d}(i_2i_3)$. When h is odd: $\sum_{i_0=0}^{2d-1} \omega^{-\frac{d}{2}hi_0} = 0$, otherwise for h even: $\sum_{i_2=0}^{d/2-1} \omega^{-hi_2} = 0$ for $h \neq 0$, similarly when v is odd: $\sum_{i_1=0}^{2d-1} \omega^{-\frac{d}{2}vi_1} = 0$, otherwise for v even: $\sum_{i_3=0}^{d/2-1} \omega^{-vi_3} = 0$ for $v \neq 0$.

Now consider the case when i_2, i_3 is odd. Then we have

$$\begin{aligned}
&\omega^{-\frac{d}{2}vh} \sum_{i_0=0}^{2d-1} \sum_{i_1=0}^{2d-1} \sum_{i_2=0}^{d/2-1} \sum_{i_3=0}^{d/2-1} \omega^{-\frac{d}{2}hi_0 - \frac{d}{2}vi_1 + \lfloor \mathcal{D} \rfloor - \lfloor \mathcal{D} + \frac{1}{2}(2hi_2 + h + 2vi_3 + v) \rfloor} \\
&= \omega^{-\frac{d}{2}vh - h - v} \sum_{i_0=0}^{2d-1} \sum_{i_1=0}^{2d-1} \sum_{i_2=0}^{d/2-1} \sum_{i_3=0}^{d/2-1} \omega^{-\frac{d}{2}hi_0 - \frac{d}{2}vi_1 - hi_2 - vi_3} \\
&= \omega^{-\frac{d}{2}vh - h - v} \left(\sum_{i_0=0}^{2d-1} \omega^{-\frac{d}{2}hi_0} \right) \left(\sum_{i_1=0}^{2d-1} \omega^{-\frac{d}{2}vi_1} \right) \left(\sum_{i_2=0}^{d/2-1} \omega^{-hi_2} \right) \left(\sum_{i_3=0}^{d/2-1} \omega^{-vi_3} \right),
\end{aligned}$$

where $\mathcal{D} = \frac{1}{2}(i_0(2i_3 + 1) + i_1(2i_2 + 1)) + \frac{1}{2d}(2i_2 + 1)(2i_3 + 1)$. Which as before, is zero. So \mathbf{S} satisfies the second condition of the GAOP. Thus \mathbf{S} is perfect. \blacksquare

The following construction is an n -dimensional generalization of Construction V.

CONSTRUCTION VI Let

$$\mathbf{S}' = [S'_{i_0, i_1, \dots, i_{4m-1}}] = \omega \left\lfloor \frac{\sum_{n=0}^{m-1} (d i_n + i_{n+2m})(d i_{n+m} + i_{n+3m})}{2d} \right\rfloor$$

be a $4m$ -dimensional array of size $\overbrace{2d \times 2d \times \dots \times 2d}^{2m \text{ terms}} \times \overbrace{d \times d \times \dots \times d}^{2m \text{ terms}}$, where $\omega = e^{2\pi \sqrt{-1}/d}$ and d is even. Let

$$\mathbf{S} = [S_{i_0, i_1, \dots, i_{2m-1}}] = \omega \left\lfloor \frac{\sum_{n=0}^{m-1} i_n i_{n+m}}{2d} \right\rfloor$$

be a $2m$ -dimensional array of size $2d^2 \times 2d^2 \times \dots \times 2d^2$ formed by concatenating the array \mathbf{S}' .

Theorem 4.8. *The array \mathbf{S} from Construction VI is perfect.*

Proof. We show the array \mathbf{S} is perfect by showing it has the GAOP for the divisor d . Firstly, we show that all distinct m -dimensional arrays of \mathbf{S}' are orthogonal.

$$\begin{aligned} & \theta_{\mathbf{S}'[i_{2m}, i_{2m+1}, \dots, i_{4m-1}], \mathbf{S}'[i'_{2m}, i'_{2m+1}, \dots, i'_{4m-1}]}(s_0, s_1, \dots, s_{2m-1}) = \\ & \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{2m-1}=0}^{d-1} S'_{i_0, i_1, \dots, i_{4m-1}} S'^*_{i_0+s_0, i_1+s_1, \dots, i_{2m-1}+s_{2m-1}, i'_{2m}, i'_{2m+1}, \dots, i'_{4m-1}} \\ & = \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{2m-1}=0}^{d-1} \left(\omega \left\lfloor \frac{\sum_{n=0}^{m-1} (d i_n + i_{n+2m})(d i_{n+m} + i_{n+3m})}{2d} \right\rfloor \times \right. \\ & \quad \left. \omega \left\lfloor \frac{\sum_{n=0}^{m-1} (d(i_n + s_n) + i'_{n+2m})(d(i_{n+m} + s_{n+m}) + i'_{n+3m})}{2d} \right\rfloor \right) \\ & = \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{2m-1}=0}^{d-1} \omega \left\lfloor \frac{\sum_{n=0}^{m-1} \mathcal{A}}{2d} \right\rfloor \omega \left\lfloor \frac{\sum_{n=0}^{m-1} \mathcal{B}}{2d} \right\rfloor, \end{aligned}$$

where $\mathcal{A} = d^2 i_n i_{m+n} + d i_{m+n} i_{2m+n} + d i_n i_{3m+n} + i_{2m+n} i_{3m+n}$ and $\mathcal{B} = d^2 s_n i_{m+n} + d^2 i_n s_{m+n} + d^2 i_n i_{m+n} + d^2 s_n s_{m+n} + d i'_{2m+n} i_{m+n} + d i'_{3m+n} i_n + d s_n i'_{3m+n} + d s_{m+n} i'_{2m+n} + i'_{2m+n} i'_{3m+n}$. We split the summation

above into $i_0, i_1, \dots, i_{2m-1}$ even and odd. Consider the case when $i_0, i_1, \dots, i_{2m-1}$ is even. Then we have

$$\begin{aligned}
& \sum_{i_0=0}^{d/2-1} \sum_{i_1=0}^{d/2-1} \dots \sum_{i_{2m-1}=0}^{d/2-1} \omega^{\sum_{n=0}^{m-1} (2di_n i_{m+n} + i_{2m+n} i_{m+n} + i_n i_{3m+n}) + \left\lfloor \frac{\sum_{n=0}^{m-1} \mathcal{C}}{2d} \right\rfloor} \times \\
& \quad \omega^{-\sum_{n=0}^{m-1} (ds_n i_{m+n} + di_n s_{m+n} + 2di_n i_{m+n} + i_{m+n} i'_{2m+n} + i'_{3m+n} i_n) - \left\lfloor \frac{\sum_{n=0}^{m-1} \mathcal{D}}{2d} \right\rfloor} \\
&= \sum_{i_0=0}^{d/2-1} \sum_{i_1=0}^{d/2-1} \dots \sum_{i_{2m-1}=0}^{d/2-1} \omega^{-\sum_{n=0}^{m-1} ((i_{3m+n} - i'_{3m+n} - ds_{m+n}) i_n + (i_{2m+n} - i'_{2m+n} - ds_n) i_{m+n}) + \left\lfloor \frac{\sum_{n=0}^{m-1} \mathcal{C}}{2d} \right\rfloor - \left\lfloor \frac{\sum_{n=0}^{m-1} \mathcal{D}}{2d} \right\rfloor} \\
&= \omega^{\left\lfloor \frac{\sum_{n=0}^{m-1} \mathcal{C}}{2d} \right\rfloor - \left\lfloor \frac{\sum_{n=0}^{m-1} \mathcal{D}}{2d} \right\rfloor} \sum_{i_0=0}^{d/2-1} \sum_{i_1=0}^{d/2-1} \dots \sum_{i_{2m-1}=0}^{d/2-1} \omega^{-\sum_{n=0}^{m-1} ((i_{3m+n} - i'_{3m+n} - ds_{m+n}) i_n + (i_{2m+n} - i'_{2m+n} - ds_n) i_{m+n})} \\
&= \omega^{\left\lfloor \frac{\sum_{n=0}^{m-1} \mathcal{C}}{2d} \right\rfloor - \left\lfloor \frac{\sum_{n=0}^{m-1} \mathcal{D}}{2d} \right\rfloor} \left(\sum_{i_0=0}^{d/2-1} \omega^{(i_{3m} - i'_{3m} - ds_m) i_0} \right) \left(\sum_{i_1=0}^{d/2-1} \omega^{(i_{3m+1} - i'_{3m+1} - ds_{m+1}) i_1} \right) \times \dots \times \\
& \quad \left(\sum_{i_{m-1}=0}^{d/2-1} \omega^{(i_{4m-1} - i'_{4m-1} - ds_{2m-1}) i_{m-1}} \right) \times \left(\sum_{i_m=0}^{d/2-1} \omega^{(i_{2m} - i'_{2m} - ds_0) i_m} \right) \times \\
& \quad \left(\sum_{i_{m+1}=0}^{d/2-1} \omega^{(i_{2m+1} - i'_{2m+1} - ds_1) i_{m+1}} \right) \times \dots \times \left(\sum_{i_{2m-1}=0}^{d/2-1} \omega^{(i_{3m-1} - i'_{3m-1} - ds_{m-1}) i_{2m-1}} \right),
\end{aligned}$$

where $\mathcal{C} = i_{2m+n} i_{3m+n}$ and $\mathcal{D} = d^2 s_n s_{m+n} + ds_n i'_{3m+n} + ds_{m+n} i'_{2m+n} + i'_{2m+n} i'_{3m+n}$. All the Gaussian sums above are zero as $i_k \neq i'_k$, for all k . Now consider the case where $i_0, i_1, \dots, i_{2m-1}$ is odd. In this case we have $\mathcal{C} = d^2 + di_{2m+n} + di_{3m+n} + i_{2m+n} i_{3m+n}$ and $\mathcal{D} = d^2 s_n s_{m+n} + d^2 s_{m+n} + d^2 s_n + d^2 + ds_n i'_{3m+n} + di'_{2m+n} s_{m+n} + di'_{2m+n} + di'_{3m+n} + i'_{2m+n} i'_{3m+n}$, but the product of Gaussian sums is the same as the case above. Thus, \mathbf{S} satisfies the first condition of the GAOP.

We now show \mathbf{S} satisfies the second condition of the GAOP.

$$\begin{aligned}
& \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{2m-1}=0}^{d-1} \theta_{\mathbf{S}'[i_{2m}, i_{2m+1}, \dots, i_{4m-1}]}(s_0, s_1, \dots, s_{2m-1}) = \\
& \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{4m-1}=0}^{d-1} S_{i_0, i_1, \dots, i_{4m-1}} S_{i_0+s_0, i_1+s_1, \dots, i_{2m-1}+s_{2m-1}, i_{2m}, i_{2m+1}, \dots, i_{4m-1}}^* \\
&= \sum_{i_0=0}^{d-1} \sum_{i_1=0}^{d-1} \dots \sum_{i_{4m-1}=0}^{d-1} \omega^{\left\lfloor \frac{\sum_{n=0}^{m-1} (d(i_n + i_{n+2m})(i_n + m + i_{n+3m}))}{2d} \right\rfloor} \times \\
& \quad \omega^{-\left\lfloor \frac{\sum_{n=0}^{m-1} (d(i_n + s_n) + i_{n+2m})(d(i_n + m + s_n + m) + i_{n+3m}))}{2d} \right\rfloor}.
\end{aligned}$$

We split the summation above into $i_0, i_1, \dots, i_{4m-1}$ even and odd. Consider the case when $i_0, i_1, \dots, i_{4m-1}$ is even. Then we have

$$\sum_{i_0=0}^{d/2-1} \sum_{i_1=0}^{d/2-1} \dots \sum_{i_{4m-1}=0}^{d/2-1} \omega^{-\sum_{n=0}^{m-1} (ds_{m+n} i_n + ds_n i_{m+n} + s_{m+n} i_{2m+n} + s_n i_{3m+n}) + \lfloor \mathcal{A} \rfloor - \lfloor \mathcal{A} + \frac{1}{2} \sum_{n=0}^{m-1} ds_n s_{m+n} \rfloor},$$

where $\mathcal{A} = \frac{1}{d} \sum_{n=0}^{m-1} 2i_{2m+n} i_{3m+n}$. As d is even, $\lfloor \mathcal{A} + \frac{1}{2} \sum_{n=0}^{m-1} ds_n s_{m+n} \rfloor = \lfloor \mathcal{A} \rfloor + \frac{1}{2} \sum_{n=0}^{m-1} ds_n s_{m+n}$, then

the summation above becomes

$$\begin{aligned}
& \omega^{-\frac{1}{2} \sum_{n=0}^{m-1} ds_n s_{m+n}} \sum_{i_0=0}^{d/2-1} \sum_{i_1=0}^{d/2-1} \dots \sum_{i_{4m-1}=0}^{d/2-1} \omega^{-\sum_{n=0}^{m-1} (ds_{m+n} i_n + ds_n i_{m+n} + s_{m+n} i_{2m+n} + s_n i_{3m+n})} \\
&= \omega^{-\frac{1}{2} \sum_{n=0}^{m-1} ds_n s_{m+n}} \sum_{i_0=0}^{d/2-1} \sum_{i_1=0}^{d/2-1} \dots \sum_{i_{4m-1}=0}^{d/2-1} \omega^{-\sum_{n=0}^{m-1} (ds_{m+n} i_n + ds_n i_{m+n})} \omega^{-\sum_{n=0}^{m-1} (s_{m+n} i_{2m+n} + s_n i_{3m+n})} \\
&= \omega^{-\frac{1}{2} \sum_{n=0}^{m-1} ds_n s_{m+n}} \left(\sum_{i_0=0}^{d/2-1} \sum_{i_1=0}^{d/2-1} \dots \sum_{i_{2m-1}=0}^{d/2-1} \omega^{-\sum_{n=0}^{m-1} (ds_{m+n} i_n + ds_n i_{m+n})} \right) \times \\
&\quad \left(\sum_{i_{2m}=0}^{d/2-1} \sum_{i_{2m+1}=0}^{d/2-1} \dots \sum_{i_{4m-1}=0}^{d/2-1} \omega^{-\sum_{n=0}^{m-1} (s_{m+n} i_{2m+n} + s_n i_{3m+n})} \right) \\
&= \omega^{-\frac{1}{2} \sum_{n=0}^{m-1} ds_n s_{m+n}} \left(\sum_{i_0=0}^{d/2-1} \sum_{i_1=0}^{d/2-1} \dots \sum_{i_{2m-1}=0}^{d/2-1} \omega^{-\sum_{n=0}^{m-1} (ds_{m+n} i_n + ds_n i_{m+n})} \right) \times \\
&\quad \left(\sum_{i_{2m}=0}^{d/2-1} \sum_{i_{2m+1}=0}^{d/2-1} \dots \sum_{i_{3m-1}=0}^{d/2-1} \omega^{-\sum_{n=0}^{m-1} s_{m+n} i_{2m+n}} \right) \left(\sum_{i_{3m}=0}^{d/2-1} \sum_{i_{3m+1}=0}^{d/2-1} \dots \sum_{i_{4m-1}=0}^{d/2-1} \omega^{-\sum_{n=0}^{m-1} s_n i_{3m+n}} \right) \\
&= \omega^{-\frac{1}{2} \sum_{n=0}^{m-1} ds_n s_{m+n}} \left(\sum_{i_0=0}^{d/2-1} \sum_{i_1=0}^{d/2-1} \dots \sum_{i_{2m-1}=0}^{d/2-1} \omega^{-\sum_{n=0}^{m-1} (ds_{m+n} i_n + ds_n i_{m+n})} \right) \times \\
&\quad \left(\sum_{i_{2m}=0}^{d/2-1} \omega^{-s_m i_{2m}} \right) \left(\sum_{i_{2m+1}=0}^{d/2-1} \omega^{-s_{m+1} i_{2m+1}} \right) \times \dots \times \left(\sum_{i_{3m-1}=0}^{d/2-1} \omega^{-s_{2m-1} i_{3m-1}} \right) \times \\
&\quad \left(\sum_{i_{3m}=0}^{d/2-1} \omega^{-s_0 i_{3m}} \right) \left(\sum_{i_{3m+1}=0}^{d/2-1} \omega^{-s_1 i_{3m+1}} \right) \times \dots \times \left(\sum_{i_{4m-1}=0}^{d/2-1} \omega^{-s_{m-1} i_{4m-1}} \right).
\end{aligned}$$

As we are computing off-peak autocorrelations, at least one of the Gaussian sums above is zero. A similar calculation shows the summation is zero for $i_0, i_1, \dots, i_{4m-1}$ odd. So, \mathbf{S} satisfies the second condition of the GAOP. Thus, \mathbf{S} is perfect. \blacksquare

It is currently unknown if constructions similar to Construction V and Construction VI exist for d odd. The arrays \mathbf{S}' and \mathbf{S} in Constructions V and VI are not perfect for d odd.

The following construction is a multi-dimensional generalization of Milewski's sequence construction.

CONSTRUCTION VII Let $\mathbf{u} = [u_0, u_1, \dots, u_{r-1}]$ be a Chu sequence [Chu, 1972], then let

$$\mathbf{S}' = [S'_{i_0, i_1, \dots, i_{2m-1}}] = \left(\prod_{n=0}^{m-1} u_{i_n} \right) \omega^{\left(\prod_{n=m}^{2m-1} i_n + \sum_{n=0}^{m-1} i_n i_{n+m} \right)}$$

be a $2m$ -dimensional array of size $\overbrace{r^{k+1} \times \dots \times r^{k+1}}^{m \text{ terms}} \times \overbrace{r^k \times \dots \times r^k}^{m \text{ terms}}$, where $\omega = e^{2\pi\sqrt{-1}/r^{k+1}}$ and r is even. Let \mathbf{S} be the m -dimensional array of size $r^{2k+1} \times r^{2k+1} \times \dots \times r^{2k+1}$ formed by concatenating the array \mathbf{S}' .

Theorem 4.9. *The array \mathbf{S} from Construction VII is perfect.*

Proof. We show the array \mathbf{S} is perfect by showing it has the GAOP for the divisor r^k . Firstly, we show that all distinct m -dimensional arrays of \mathbf{S}' are orthogonal.

$$\begin{aligned}
& \theta_{\mathbf{S}'[i_m, i_{m+1}, \dots, i_{2m-1}], \mathbf{S}'[i'_m, i'_{m+1}, \dots, i'_{2m-1}]}(s_0, s_1, \dots, s_{m-1}) = \\
& \sum_{i_0=0}^{r^{k+1}-1} \sum_{i_1=0}^{r^{k+1}-1} \cdots \sum_{i_{m-1}=0}^{r^{k+1}-1} S'_{i_0, i_1, \dots, i_{2m-1}} S'^{*}_{i_0+s_0, i_1+s_1, \dots, i_{m-1}+s_{m-1}, i'_m, i'_{m+1}, \dots, i'_{2m-1}} \\
& = \sum_{i_0=0}^{r^{k+1}-1} \sum_{i_1=0}^{r^{k+1}-1} \cdots \sum_{i_{m-1}=0}^{r^{k+1}-1} \left(\left(\prod_{n=0}^{m-1} u_{i_n} \right) \omega^{\prod_{n=m}^{2m-1} i_n + \sum_{n=0}^{m-1} i_n i_{n+m}} \right) \left(\left(\prod_{n=0}^{m-1} u_{i_n+s_n} \right) \omega^{\prod_{n=m}^{2m-1} i'_n + \sum_{n=0}^{m-1} (i_n+s_n) i'_{n+m}} \right)^* \\
& = \omega^{\prod_{n=m}^{2m-1} i_n - \prod_{n=m}^{2m-1} i'_n - \sum_{n=0}^{m-1} s_n i_{n+m}} \sum_{i_0=0}^{r^{k+1}-1} \sum_{i_1=0}^{r^{k+1}-1} \cdots \sum_{i_{m-1}=0}^{r^{k+1}-1} \left(\left(\prod_{n=0}^{m-1} u_{i_n} u_{i_n+s_n}^* \right) \omega^{-\sum_{n=0}^{m-1} (i_{n+m} - i'_{n+m}) i_n} \right) \\
& = \omega^{\prod_{n=m}^{2m-1} i_n - \prod_{n=m}^{2m-1} i'_n - \sum_{n=0}^{m-1} s_n i_{n+m}} \left(\sum_{i_0=0}^{r^{k+1}-1} \sum_{i_1=0}^{r^{k+1}-1} \cdots \sum_{i_{m-1}=0}^{r^{k+1}-1} \prod_{n=0}^{m-1} u_{i_n} u_{i_n+s_n}^* \omega^{(i'_{n+m} - i_{n+m}) i_n} \right) \\
& = \omega^{\prod_{n=m}^{2m-1} i_n - \prod_{n=m}^{2m-1} i'_n - \sum_{n=0}^{m-1} s_n i_{n+m}} \left(\prod_{n=0}^{m-1} \sum_{i_n=0}^{r^{k+1}-1} u_{i_n} u_{i_n+s_n}^* \omega^{(i'_{n+m} - i_{n+m}) i_n} \right)
\end{aligned}$$

As r is even, a term in the Chu sequence is given by $u_n = e^{\left(\frac{\pi\sqrt{-1}}{r}\right)pn^2}$, where p is relatively prime to r^{k+1} . Then we have: $\sum_{i_n=0}^{r^{k+1}-1} u_{i_n} u_{i_n+s_n}^* \omega^{(i'_{n+m} - i_{n+m}) i_n} = \sum_{i_n=0}^{r^{k+1}-1} e^{\left(\frac{2\pi\sqrt{-1}}{r^{k+1}}\right)\left(\frac{r^k}{2}pi_n^2 - \frac{r^k}{2}p(i_n+s_n)^2 + i'_{n+m} - i_{n+m}\right)i_n} = e^{\left(\frac{2\pi\sqrt{-1}}{r^{k+1}}\right)\left(-\frac{1}{2}pr^ks_n^2\right)} \sum_{i_n=0}^{r^{k+1}-1} e^{\left(\frac{2\pi\sqrt{-1}}{r^{k+1}}\right)\left(-r^kps_n + i'_{n+m} - i_{n+m}\right)i_n} = 0$ as $i'_{n+m} \neq i_{n+m}$. So \mathbf{S} satisfies the first condition of the GAOP. We now show \mathbf{S} satisfies the second condition of the GAOP.

$$\begin{aligned}
& \sum_{i_0=0}^{r^{k+1}-1} \cdots \sum_{i_{m-1}=0}^{r^{k+1}-1} \theta_{\mathbf{S}'[i_m, i_{m+1}, \dots, i_{2m-1}]}(s_0, s_1, \dots, s_{m-1}) = \\
& \sum_{i_0=0}^{r^{k+1}-1} \cdots \sum_{i_{m-1}=0}^{r^{k+1}-1} \sum_{i_m=0}^{r^k-1} \cdots \sum_{i_{2m-1}=0}^{r^k-1} S'_{i_0, i_1, \dots, i_{2m-1}} S'^*_{i_0+s_0, i_1+s_1, \dots, i_{m-1}+s_{m-1}, i_m, i_{m+1}, \dots, i_{2m-1}} \\
& = \sum_{i_0=0}^{r^{k+1}-1} \cdots \sum_{i_{m-1}=0}^{r^{k+1}-1} \sum_{i_m=0}^{r^k-1} \cdots \sum_{i_{2m-1}=0}^{r^k-1} \left(\left(\prod_{n=0}^{m-1} u_{i_n} \right) \omega^{(\prod_{n=m}^{2m-1} i_n + \sum_{n=0}^{m-1} i_n i_{n+m})} \right) \times \\
& \quad \left(\left(\prod_{n=0}^{m-1} u_{i_n+s_n} \right) \omega^{(\prod_{n=m}^{2m-1} i_n + \sum_{n=0}^{m-1} (i_n+s_n) i_{n+m})} \right)^* \\
& = \sum_{i_0=0}^{r^{k+1}-1} \cdots \sum_{i_{m-1}=0}^{r^{k+1}-1} \sum_{i_m=0}^{r^k-1} \cdots \sum_{i_{2m-1}=0}^{r^k-1} \left(\left(\prod_{n=0}^{m-1} u_{i_n} u_{i_n+s_n}^* \right) \omega^{-\sum_{n=0}^{m-1} s_n i_{n+m}} \right) \\
& = \sum_{i_0=0}^{r^{k+1}-1} \cdots \sum_{i_{m-1}=0}^{r^{k+1}-1} \sum_{i_m=0}^{r^k-1} \cdots \sum_{i_{2m-1}=0}^{r^k-1} \left(\left(\prod_{n=0}^{m-1} u_{i_n} u_{i_n+s_n}^* \right) \left(\prod_{n=0}^{m-1} \omega^{-s_n i_{n+m}} \right) \right) \\
& = \left(\sum_{i_0=0}^{r^{k+1}-1} \cdots \sum_{i_{m-1}=0}^{r^{k+1}-1} \left(\prod_{n=0}^{m-1} u_{i_n} u_{i_n+s_n}^* \right) \right) \left(\sum_{i_m=0}^{r^k-1} \cdots \sum_{i_{2m-1}=0}^{r^k-1} \left(\prod_{n=0}^{m-1} \omega^{-s_n i_{n+m}} \right) \right) \\
& = \left(\prod_{n=0}^{m-1} \sum_{i_n=0}^{r^k-1} u_{i_n} u_{i_n+s_n}^* \right) \left(\prod_{n=0}^{m-1} \sum_{i_n=0}^{r^k-1} \omega^{-s_n i_{n+m}} \right)
\end{aligned}$$

For $s_n \neq 0 \pmod r$, $\sum_{i_n=0}^{r^k-1} u_{i_n} u_{i_n+s_n}^* = 0$, otherwise for $s_n = 0 \pmod r$ and $s_n \neq 0$, $\sum_{i_n=0}^{r^k-1} \omega^{-s_n i_{n+m}} = 0$. So \mathbf{S} satisfies the second condition of the GAOP. Thus \mathbf{S} is perfect. \blacksquare

Finally, we note that the array \mathbf{S}' from Construction VII is perfect for odd r .

Theorem 4.10. *The array \mathbf{S}' from Construction VII is perfect for odd r .*

Proof. The proof is similar to those given for the previous constructions. \blacksquare

5 Conclusions

We have generalised the AOP to higher dimensions and showed that a n -dimensional array with the GAOP is perfect. Using the GAOP, we have derived a number of perfect array constructions. Each of these array constructions are bounded in size. It is unknown if there exist array constructions with the GAOP which are unbounded in size.

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Appendix I – Implementation of the Constructions

In this appendix we show the implementations of the n -dimensional constructions in the computer algebra system, Mathematica (version 8.0). (All arrays are given in index notation, that is, the mapping: $e^{2\pi\sqrt{-1}s_n/r} \rightarrow s_n$.)

We begin with the code for periodic cross-correlation, **XCV** and autocorrelation, **ACV**:

```
In[1]:= XCV[a_, b_, r_Integer] := Block[{A, B},
  A = Developer`ToPackedArray[ Exp[(2. Pi I a)/r] ];
  B = Developer`ToPackedArray[ Exp[(-2. Pi I b)/r] ];
  Chop[ListCorrelate[A, B, 1], 1*^-5]]
```

```
In[2]:= ACV[m_, r_Integer] := XCV[m, m, r]
```

The function **index** takes as input the dimension, d , and returns the index function for the d -dimensional array, $\prod_{n=d/2}^{d-1} i_n + \sum_{n=0}^{d/2-1} i_n i_{n+d/2}$ (note that in Mathematica, array indexing starts at 1):

```
In[1]:= index[d_?EvenQ] :=
  Function @@ {Sum[Slot[n] Slot[n + d/2], {n, d/2}] + Product[Slot[n], {n, d/2 + 1, d}]}
```

For example, we compute the index function for the 2-dimensional array from Construction IV:

```
In[2]:= index[2]
Out[2]= #1 + #1 #2 &
```

And now the index function for the 8-dimensional array:

```
In[3]:= index[8]
Out[3]= #1 #5 + #2 #6 + #3 #7 + #4 #8 + #4 #5 #6 #7 #8 &
```

The following function implements Construction IV. It takes as inputs the number of roots of unity, **nr**, and the number of dimensions, **nd** and returns the multi-dimensional perfect array, **S**:

```
In[4]:= ConstructionVI[nr_Integer, nd_?EvenQ] := With[{indexF = index[2 nd]},
  ArrayFlatten[Mod[Array[indexF, Table[nr, {2 nd}]], nr], nd]]
```

For example, the following is a perfect 4-dimensional binary array:

```
In[5]:= ConstructionVI[2, 4]
```

```
Out[5]=
```

$$\left[\begin{array}{c} \left[\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right] \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right] \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right] \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right] \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \\ \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right] \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{array} \right] \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{array} \right] \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right] \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \end{array} \right]$$

The following code snippet gives the full calculation from Example 4.5. We begin by showing \mathbf{S} satisfies the first condition of the GAOP. The following code generates all arrays $\mathbf{S}'[i, j]$, for $0 \leq i < 3$ and $0 \leq j < 3$:

```
In[6]:= allSps = Join @@ Table[
  Table[
    ConstructionVI[3, 2][[3 n + 1 + q, 3 m + 1 + r]],
    {n, 0, 2}, {m, 0, 2}],
  {q, 0, 2}, {r, 0, 2}];
```

We now generate all 36 distinct pairs of arrays, compute their cross correlation, and count the number of zeros in the resulting array of cross-correlation values:

```
In[7]:= Count[XCV[#1, #2, 3], 0, {2}] & @@@
  Union[ Select[Sort /@ Tuples[allSps, 2], First[#] != Last[#] & ] ]

Out[7]= {9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, \
  9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9}
```

We see that for each pair of arrays there are 9 zero cross-correlation values. Thus, each pair of arrays are orthogonal. We now show \mathbf{S} satisfies the second condition of the GAOP. For each array, we compute its autocorrelation for all shifts. We then sum all the autocorrelations together.

```
In[8]:= Chop @ Total[ACV[#, 3] & /@ allSps]

Out[8]= {{81, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

So for all off-peak shifts, the sum of the autocorrelation is zero. Thus, \mathbf{S} satisfies the second condition of the GAOP.

The following function implements Construction V. It takes as input the number of roots, d , and returns the perfect array \mathbf{S} :

```
In[9]:= ConstructionVII[d_?EvenQ] := Mod[Array[Floor[#1 #2/(2 d)] &, {2 d^2, 2 d^2}], d]
```

For example, the following is a perfect binary array:

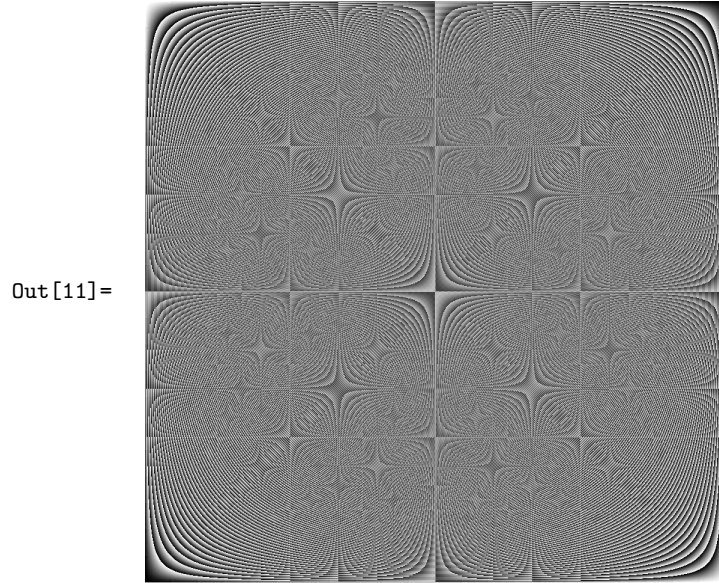
```
In[10]:= ConstructionVII[2]
```


Out[10]=

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

These arrays are highly symmetric and exhibit a beautiful structure. The following is a perfect 968×968 perfect array over 22 roots of unity:

In[11]:= ArrayPlot[ConstructionVII[22], Frame -> False]



The following function implements Construction VI. It takes as inputs the number of roots of unity, nr , and the number of dimensions, nd and returns the multi-dimensional perfect array, S :

In[12]:= indexVIII[d_?EvenQ] := Function @@ {Sum[Slot[n] Slot[n + d/2], {n, d/2}]}

In[13]:= ConstructionVIII[nr_Integer, nd_?EvenQ] := With[{indexP = indexVIII[nd][[1]]},
Array[Mod[Floor[indexP/(2 nr)], nr] &, Table[2 nr^2, {nd}]]]

The following function implements Construction VII. It takes as input the number of roots of unity, nr , the k parameter, k , and the number of dimensions, nd and returns the multi-dimensional perfect array, S :

In[14]:= indexIX[nr_?EvenQ, k_Integer, nd_?EvenQ] :=
Function @@ {Sum[Slot[n] (Slot[n] + 1) nr^k/2, {n, 1, nd/2}] + index[nd][[1]]}

In[15]:= indexIX[nr_?OddQ, k_Integer, nd_?EvenQ] :=
Function @@ {Sum[Slot[n] (Slot[n] + 1) nr^k, {n, 1, nd/2}] + index[nd][[1]]}

In[16]:= ConstructionIX[nr_Integer, k_Integer, nd_?EvenQ] := With[{indexF = indexIX[nr, k, nd]},
ArrayFlatten[
Mod[Array[indexF, Table[nr^(k + 1), {nd/2}]]~Join~Table[nr^k, {nd/2}], 0],
nr^(k + 1), nd/2]]

The array plot below shows the beautiful structure of a perfect 1024×1024 array over 64 roots of unity:

In[17]:= ArrayPlot[ConstructionIX[4, 2, 4], Frame -> False]

Out[17]=

